Universität Heidelberg - WS22/23

Talk 5 -Morphisms of sheaves

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Contents

Introduction

The aim of this talk was to establish the categories of presheaves and sheaves and to show that those are abelian categories. The definitions and results of the last talks are stated in the language of category theory and the notions of kernels and cokernels as well as exact sequences are introduced.

For the entire transcript let X a topological space. Furthermore all sheaves and presheaves are considered as sheaves and presheaves of abelian groups over X.

The categories $\mathbf{Presh}_X$ and $\mathbf{Shv}_X$

To remind the definition of basic category-theoretical terms and some definitions from the last talks we give a few examples.

Examples 1. (Categories)

- $\mathbf{Presh}_X$: The category of presheaves over X with the presheaves over X as objects and presheaf morphisms as morphisms. For F, G presheaves a presheaf morphism $f : F \to G$ is given by maps $f(U) : F(U) \to G(U)$ for all $U \subseteq X$ such that for each $V \subseteq U$ open in X the following diagram commutes:

$$\begin{array}{ccc} F(U) & \xrightarrow{f(U)} & G(U) \\ \downarrow{\scriptstyle \rho_V^U} & & \downarrow{\scriptstyle \rho'^U_V} \\ F(V) & \xrightarrow{f(V)} & G(V) \end{array}$$

Note that the restriction ρ_V^U on the left is the restriction map of F and the restriction ρ'^U_V on the right is the restriction map of G

- $\mathbf{Shv}_X$: The category of sheaves over X is denoted as $\mathbf{Shv}_X$. Its objects are the sheaves over X and the presheaf morphisms between sheaves are the morphisms.

Therefore $\mathbf{Shv}_X$ is a full subcategory of $\mathbf{Presh}_X$

- The category of sheafspaces over X: $\mathbf{Shsp}_X$

- A preordered set Λ can be seen as a category:
The objects of this category are the elements of Λ and for two objects μ, λ we define

$$\mathrm{Hom}(\mu, \lambda) = \begin{cases} \text{singleton} & \mu \leq \lambda \\ \emptyset & \text{else} \end{cases}$$

Examples 2. (Functors)

- The inclusion functor $\mathbf{Shv}_X \to \mathbf{Presh}_X$ which sends each sheaf to its underlying presheaf.
- $L : \mathbf{Presh}_X \to \mathbf{Shsp}_X$ from last talk
- $\Gamma : \mathbf{Shsp}_X \to \mathbf{Shv}_X$ from last talk. For these we checked the conditions for functoriality in the last talk.
- As the composition of two functors is again a functor, we get a functor $\Gamma L : \mathbf{Presh}_X \to \mathbf{Shv}_X$ which is called the "sheafification functor"
- The functor $\Gamma(-, U) : \mathbf{Shv}_X \to \mathbf{Ab}$, $F \mapsto F(U)$ for an open set $U \subseteq X$.
- Presheaves themselves can be seen as functors:
  Let $\mathcal{U}$ the set of open subsets of X. $\mathcal{U}$ is preordered by "$\subseteq$". Hence we can view $\mathcal{U}$ as a category. A presheaf corresponds to a functor $\mathcal{U}^{\text{op}} \to \mathbf{Ab}$.

$\mathcal{U}$ denotes the dual category of $\mathcal{U}$

Examples 3. (Natural transformations)

- The functor $L\Gamma$ is naturally equivalent to $id_{\mathbf{Shsp}_X}$.
- There is a natural transformation $id_{\mathbf{Presh}_X} \Rightarrow \Gamma L$ and $\Gamma L_{|\mathbf{Shv}_X} \cong id_{\mathbf{Shv}_X}$.
- presheaf morphisms can be expresses as natural transformations:
  For two presheaves F, G understood as functors by the last example the condition of naturality of η is given by

To be more precise using terms of category theory: ΓL is left adjoint to the inclusion functor

$$\begin{CD} F(U) @>{\eta(U)}>> G(U) \\ @V{F(i_{V\to U})}VV @VV{G(i_{V\to U})}V \\ F(V) @>{\eta(V)}>> G(V) \end{CD}$$

This corresponds with the definition given in **??**

Remark 4. From the last example we get an alternate description of the category of presheaves over X:

$$\mathbf{Presh}_X = Fun(\mathcal{U}^{\text{op}}, \mathbf{Ab})$$

Next our aim is to show that $\mathbf{Presh}_X$ and $\mathbf{Shv}_X$ are abelian categories.

Definition 5. (Preadditive category)
A category $\mathcal{C}$ is called preadditive if for each $A, B \in Ob\mathcal{C}$ the set $\text{Hom}_{\mathcal{C}}(A, B)$ is an abelian group and composition is linear, i.e. for $f, g : A \to B$ and $h : B \to C$ the composition $h \circ (f + g) = h \circ f + h \circ g$ and for $f : A \to B$ and $g, h : B \to C$ analogously $(g + h) \circ f = g \circ f + h \circ f$.

We want to construct such a group structure on the sets of presheaf morphisms:

Definitions 6. Let $f, g : F \to G$ presheaf morphisms with $F, G \in \mathbf{Presh}_X$. For all $U \subseteq X$ open we define:

Note that $G(U)$ is an abelian group, hence addition, zero and negatives are well defined in $G(U)$

- $(f+g)(U) : F(U) \to G(U)$, $s \mapsto f(s) + g(s)$
- $0(U) : F(U) \to G(U)$, $s \mapsto 0 \in G(U)$
- $-f(U) : F(U) \to G(U)$, $s \mapsto -f(s)$

Those are well defined presheaf morphisms. Therefore we have a well defined group structure on the morphism sets $\mathrm{Hom}_{\mathbf{Presh}_X}(F, G)$

Definition 7. The zero sheaf 0 is the sheaf $\{0\}_X$, i.e. $0(U)$ is a trivial abelian group for all $U \subseteq X$ open.

Remark 8. Obviously $\mathrm{Hom}(F, 0)$ and $\mathrm{Hom}(0, F)$ are trivial abelian groups too. Therefore the zero sheaf is a zero object in the categories $\mathbf{Presh}_X$ and $\mathbf{Shv}_X$.

As last part of this section we want to establish an example of a sheaf morphism, that will lead us through the rest of this talk.

Example 9. (The exponential sheaf morphism)
Let $\Omega \subseteq \mathbb{C}$ open, $\mathcal{O}_\Omega$ the sheaf of $\mathbb{C}$-valued holomorphic functions, $\mathcal{O}^*_\Omega$ the sheaf of $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$-valued holomorphic functions.
Then we can define

Note that $\mathcal{C}$ is considered as an abelian group by addition, but $\mathcal{C}^*$ as an abelian group by multiplication

$$\begin{aligned} \exp : \quad & \mathcal{O}_\Omega && \to \mathcal{O}^*_\Omega \\ \exp(U) : \quad & \mathcal{O}_\Omega(U) && \to \mathcal{O}^*_\Omega(U) \\ & f && \mapsto \exp(i2\pi f) \end{aligned}$$

This is a well defined sheaf morphism as $exp(U)$ is a group homomorphism:

$$f + g \mapsto \exp(i2\pi(f+g)) = \exp(i2\pi f) * \exp(i2\pi g)$$

Kernels and monomorphisms

The aim of this section is to define the kernel of a (pre)sheaf morphism and monomorphisms. We will see that for sheaves it doesn't matter if you take the kernel of those sheaves considered as sheaves or as presheaves. This will not be the case for cokernels later.

Definition 10. For presheaves F and G let $f : F \to G$ a presheaf morphism. Then define:

$$K(U) := \ker f(U) = s \in F(U) | f(U)(s) = 0 \tag{1}$$

This defines a sheaf as for $V \subseteq U$ open in X, $s \in K(U)$ we have:

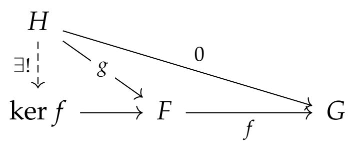
$$f(V)\rho^U_V(s) = \rho^U_V(f(U)(s)) = \rho^U_V$$

This presheaf is denoted as $\ker f$ and called the kernel of f.

Remark 11. We have a natural morphism $\ker f \to F$ and the composition $\ker f \to F \overset{f}{\Rightarrow} G$ is the zero morphism. Both of these remarks follow immediately from the definition (note that for each U $\ker f(U)$ is a subgroup of $F(U)$).

The usual universal property of kernels from general category theory holds for presheaves as well:

Proposition 12. *(Universal property of a kernel)*
For each presheaf H and presheaf morphism $g : H \to F$ s.t. $f \circ g = 0$ there is a unique morphism $H \to \ker f$ s.t. the following diagram commutes:



Proposition 13. *If F, G are sheaves, then $\ker f$ is also a sheaf.*

Proof. The monopresheaf condition is inherited directly from F.
Let $U = \cup_{\lambda \in \Lambda} U_\lambda$ and $s_\lambda \in \ker(f)(U_\lambda)$ such that the compatibility conditions are given on all $U_\lambda \cap U_\mu$. Then - as F is a sheaf - there is an $s \in F(U)$ such that $\forall \lambda \in \Lambda : \rho^U_{U_\lambda}(s) = s$. Hence for $s' = f(U)(s) \in G(U)$ the restriction $\rho^U_{U_\lambda}(s') = 0$ for all $\lambda \in \Lambda$. Using the monopreasheaf axiom for G this results in $s' = 0$ and hence $s \in \ker(f)(U)$. □

Remark 14. Actually we only used the monopresheaf axiom of G - the glueing axiom is only necessary for F.

Theorem 15. *(Characterization of monomorphisms)*
Let $f : F \to G$ a presheaf morphism. Then the following conditions are equivalent:

i. *f is an monomorphism - i.e. for two presheaf morphisms $g, h : H \to F$, $f \circ g = f \circ h$ implies $g = h$.*

ii. *$\forall U \subseteq X$ open $f(U)$ is injective*

iii. $\ker f = 0$

These imply and are equivalent if F is a monopresheaf:

iv *$\forall x \in X$ f_x is bijective.*

Proof. *i.* ⇔ *iii*. This follows from general category theory.
ii. ⇔ *iii*. This follows immediately from the definition of the kernel, as $f(U)$ is injective iff $\ker f(U) = 0$ as $f(U)$ is a group homomorphism.
ii. ⇒ *iv*. Let $t \in F_x$ s.t. $f_x(t) = 0$. We have to show that in this case already $t = 0$. There is an open set U and a section $s \in F(U)$ such that $s_x = t$. Then we have:

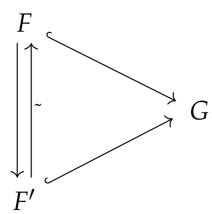
$$(f(U)(s))_x = 0$$

Again we find an open set $V \subseteq U$ such that

$$0 = \rho_V^U(f(U)(s)) = f(V)\rho_V^U(s)$$

As $f(V)$ is injective, we get $\rho_V^U = 0$ and therefore $t = 0$.
iv. $\Rightarrow$ *ii.* Now let F a monopresheaf. Let $s \in F(U)$ s.t. $f(U)(s) = 0 \in G(U)$. We have to show, that $s = 0$. For all $x \in X$ we get $f_x(s_x) = (f(U)(s))_x = 0$ and hence $s_x = 0$ as the f_x are injective. Using the monopresheaf axiom we conclude that $s = 0$. □

Definition 16. Let $f : F \hookrightarrow G$ a (pre)sheaf monomorphism with F and G (pre)sheaves. In this case we call F a sub(pre)sheaf of G or - more precisely - F represents a sub(pre)sheaf of G. Two (pre)sheaves F, F' represent the same sub(pre)sheaf of G if there is an isomorphism $F \to F'$ s.t. the followng diagram commutes:



If $F = \ker f$ as subpresheaves of G we call F a kernel of f.

Remark 17. Subobjects can be defined in arbitrary categories.

Corollary 18. *Let F a monopresheaf. Then F is a subpresheaf of a sheaf, namely its sheafification.*

Proof. In the last talk we proofed that for a presheaf F the maps $F_x \to (\Gamma LF)_x$ are isomorphisms - and therefore injective - for all $x \in X$. Using **??** *i.* $\Leftrightarrow$ *iv.* we get the corollary. □

ΓLF is the sheafification of F.

Proposition 19. *Let $f : F \to G$ a presheaf morphism.*
Then $(\ker f)_x = \ker f_x$ for all $x \in X$.

Proof.

$$\begin{aligned} t \in (\ker \mathrm{f})_x &\iff \exists \text{ open } \mathrm{U} \ni \mathrm{x} \text{ and } \mathrm{s} \in \mathrm{Ker(f)(U)} \text{ such that } \mathrm{t} = \mathrm{s_x} \\ &\iff \exists \text{ open } \mathrm{U} \ni \mathrm{x} \text{ and } \mathrm{s} \in \mathrm{F(U)} \text{ such that } t = \mathrm{s_x} \text{ and} \\ &\iff \mathrm{f_x(U)(s)} = 0 \end{aligned}$$

□

Proposition 20. *Let F, F' subsheaves of a sheaf G. Then:*

$$F = F' \quad \Leftrightarrow \quad F_x = F'_x \quad \forall x \in X$$

This last statement is given without proof - for reference see the chapter in Tennison. There a preorder for the subsheaves is defined first, which I omitted in the talk.

Lastly we go back to our example of the exponential sheaf morphism and calculate the kernel:

Example 21. For each $U \subseteq X$ open we have:

$$\begin{aligned}\ker(\exp(U)) &= \{f \in \mathcal{O}_\Omega \mid \exp(i2\pi f) = 0\} \\ &= \{f \in \mathcal{O}_\Omega \mid f \text{ is } \mathbb{Z}\text{-valued}\} \\ &= \{f : U \to \mathbb{Z},\ f \text{ locally constant}\}\end{aligned}$$

Therefore $\ker(\exp)$ is the constant sheaf $\mathbb{Z}_\Omega$.

Cokernels and epimorphisms

In contrast to kernels the cokernels are not the same in the categories of sheaves and presheaves.

Definition 22. Let $f : F \to G$ a presheaf morphism, F and G presheaves. Define:

$$C(U) := G(U) / \operatorname{Im} f(U)$$

This defines a presheaf as for the restrictions we have:

For $V \subseteq U$ open in X we have the induced restriction map

$$\rho_V^U : G(U) \to \overbrace{G(V) / \operatorname{Im} f(V)}^{C(V)}$$

Then for all $s \in F(U)$ by the naturality of the restriction maps:

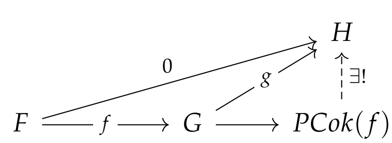
$$\rho_V^U(f(U)(s)) = f(V)\rho_V^U(s) \in \operatorname{Im} f(V)$$

Therefore the restrictions $\bar{\rho}_V^U : C(U) \to C(V)$ are well defined.

This presheaf is called the presheaf cokernel and is denoted by $PCok(f)$.

Remarks 23. There are natural morphisms $G \to PCok(f)$ and the composition $F \to G \to PCok(f)$ is the zero morphism, which follows immediately from the definition..

Proposition 24. *(Universal property)*
The universal property of the presheaf cokernel is given by:
For all presheaves H and presheaf morphisms $g : G \to H$ s.t. $g \circ h = 0$ there is a unique presheaf morphism $PCok(f) \to H$ s.t. the following diagram commutes:



Definition 25. For sheaves F and G and a sheaf morphism $F : F \to G$ the sheaf cokernel is defined as the sheafification of the presheaf cokernel:

$$SCok(F) := \Gamma L PCok(f)$$

Remark 26.
- As for the presheaf cokernel there are natural morphisms $G \to SCok(f)$ and the composition $F \to G \to SCok(f)$ is the zero morphism.
- The sheaf cokernel satisfies the same universal property as the presheaf cokernel with H a sheaf instead of a presheaf. This is an immediate consequence from the universal property of the sheafification.

Analogously to the characterization of monomorphisms in the last section we can give a characterization of epimorphisms in the categories of sheaves and presheaves.

Theorem 27. *(Characterization of presheaf epimorphisms)*
Let $f : F \to G$ a presheaf morphism. Then the following conditions are equivalent:

i $PCok(f) = 0$

ii $\forall U \subseteq X$ *open* $f(U)$ *is surjective*

iii f *is an epimorphism in* $\textbf{Presh}_X$ *- i.e. for presheaf morphisms* $g, h : G \to H$, $g \circ f = h \circ f$ *implies* $g = h$.

Proof. $i \Leftrightarrow ii$: This follows from the definition as $PCok(f)(U) = G(U)/\operatorname{Im} f(U)$ and therefore $PCok(f) = 0 \Leftrightarrow \operatorname{Im} f(U) = G(U)$.
$i \Leftrightarrow iii$: This is true for general abelian categories. □

Theorem 28. *(Characterization of sheaf epimorphisms)*
Let $f : F \to G$ a sheaf morphism. Then the following conditions are equivalent:

i $SCok(f) = 0$

ii $\forall x \in X : (PCok(f))_x = 0$

iii f_x *is surjective* $\forall x \in X$

iv f *is an epimorphism in* $\textbf{Shv}_X$.

Additionally all of the conditions from the last theorem imply these. I.e. a presheaf morphism between sheaves is automatically a sheaf morphism.

Proof. $i. \Leftrightarrow iv.$ follows from general category theory.
$i. \Leftrightarrow ii.$

$$\begin{aligned} SCok(f) = 0 &\Leftrightarrow (SCok(f))_x = 0 \quad \forall x \in X \\ &\Leftrightarrow (PCok(f))_x = 0 \quad \forall x \in X \end{aligned}$$

$ii. \Leftrightarrow iii.$

$$\begin{aligned} PCok(f)_x = 0 \Leftrightarrow &\ \forall \text{ open } U \ni x \text{ and } s \in PCok(f)(U), \\ &\quad \exists \text{ open } V \text{ with } U \supseteq V \ni x \text{ and } \rho^U_V(s) = 0 \\ \Leftrightarrow &\ \forall \text{ open } U \ni x \text{ and } s \in G(U), \\ &\quad \exists \text{ open V with } U \supseteq V \ni x \text{ and } \rho^U_V(\text{s}) \epsilon \operatorname{Im} f(\text{V}) \end{aligned}$$

$\Leftrightarrow$ f_x is surjective.

The last part of the statement is trivial as for example *i.* of the last proposition obviously implies *ii.*. ($PCokf = 0 \Rightarrow (PCok(f))_x = 0 \ \forall x \in X$) $\square$

For a sheaf morphism the presheaf cokernel is in general not the same as the sheaf cokernel. This is because the presheaf cokernel of a sheaf morphism is not always a sheaf as is seen in the following example, where we calculate the sheaf and presheaf cokernel of the esponential sheaf:

Example 29. Firstly we calculate the presheaf cokernel of exp. For $U \subseteq X$ open, we have:

$$PCok(\exp(U)) = \frac{\mathcal{O}^*_\Omega(U)}{\exp(\mathcal{O}_\Omega(U))} = \frac{\{f : U \to \mathbb{C}^* \mid f \text{ holomorphic}\}}{\{f : U \to \mathbb{C}^* \mid \exists g : U \to \mathbb{C} \text{ hol., s.t. } f = \exp(i2\pi g)\}}$$

This is in general non-empty as the exponential function is not globally invertible so for example $id_{\mathbb{C}^*}$ is a nontrivial element of $PCok(\exp(U))$ for large enough U.
On the other hand the exponential function is at least locally invertible, so $\forall x \in \mathbb{C}^*$ we find an open set $x \in U \subseteq \mathbb{C}^*$ s.t. $PCok(\exp(U)) = 0$ and therefore the stalk $(PCok(f))_x = 0$. By the last theorem then also $SCok(f) = 0$.
This means, that exp is an epimorphism in the category $\mathbf{Shv}_X$ but not in $\mathbf{Presh}_X$.

Using the earlier characterizations of monomorphisms we can characterize isomorphisms:

Corollary 30. *(Characterization of isomorphisms)*
Let $f : F \to G$ a presheaf morphism. Then the following conditions are equivalent:

i. *f is an isomorphism*

ii. *$\forall U \subseteq X$ open $f(U)$ is bijective*

iii. *f is an monomorphism and a presheaf epimorphism*

If f is a morphism of sheaves:

iv. *f is an monomorphism and a sheaf epimorphism*

v. *$\forall x \in X$ f_x is bijective.*

Proof. *i.* $\Leftrightarrow$ *iii.* and *i.* $\Leftrightarrow$ *iv.* are true in arbitrary abelian categories.
ii. $\Leftrightarrow$ *iii.* follows from the characterization of monomorphisms and epimorphisms in $\mathbf{Presh}_X$ (see **??**, **??**)
iv $\Leftrightarrow$ *v.* follows from the characterization of monomorphisms and epimorphisms in $\mathbf{Shv}_X$ (see **??**, **??**) $\square$

Proposition 31. *Let $f : F \to G$ a presheaf morphism. Then*

$$\forall x \in X : \quad (PCok(f))_x = Cok(f_x) = G_x / \operatorname{Im} f_x$$

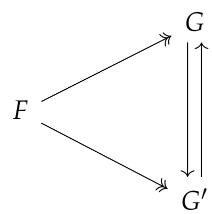
If f is a morphism of sheaves, then:

$$\forall x \in X : \quad (SCok(f))_x = Cok(f_x)$$

For a proof of this proposition see Tennison 3.4.11.

Analogously to the sub(pre)sheaf we can define quotient presheaf for epimorphisms:

Definition 32. (Quotient presheaf)
Let $f : F \to G$ a presheaf epimorphism. Then G is called a quotient presheaf of F.
Let $f : F \to G$ a sheaf epimorphism. Then G is called a quotient sheaf of F.
Two (pre)sheaves are said to represent the same quotient presheaf if there is an isomorphism s.t. the diagram:



commutes.

Theorem 33. *1. Every sub(pre)sheaf is a kernel of a (pre)sheaf morphism.*

2. Every quotient (pre)sheaf is a cokernel of a (pre)sheaf morphism.
Therfore in the categories $\textbf{Presh}_X$ *and* $\textbf{Shv}_X$ *every monomorphism is a kernel and every epimorphism is a cokernel.*

Proof. 1. Let $f : F \to G$ an (pre)sheaf monomorphism. Then $F = \ker(G \to PCok(f))$.

2. Let $f : F \to G$ an (pre)sheaf epimorphism. Then $F = PCok(\ker(f) \to F)$.

□

Therefore every condition for $\textbf{Shv}_X$ and $\textbf{Presh}_X$ to be abelian categories is checked (except for the existence of biproducts which is omitted here - see the chapter in Tennison for reference):

Definition 34. (Abelian category)
A preadditive category $\mathcal{C}$ is called abelian, if:

- $\mathcal{C}$ has a zero object. (In our case the zero sheaf)
- $\mathcal{C}$ has kernels and cokernels of all morphisms
- Every monomorphism is a kernel, every epimorphism is a cokernel in $\mathcal{C}$
- $\mathcal{C}$ has biproducts of each pair of objects - i.e. a objeact that satisfies the universal properties of products and coproducts at the same time.

There are definitions of "additive categories" found in literature that differ from Tennison - most of them already requirethe existence of biproducts for the additive category although in Tennison a additive category is only a preadditive category with zero object.

Exact sequences of (pre)sheaves

Before we can define the notion of an exact sequence of (pre)sheaves we have to define what the image of a (pre)sheaf morphism is. Unfortunately like with the cokernel the image differs in the to categories:

Definition 35. 1. Let $f : F \to G$ a presheaf morphism. The presheaf image is defined as

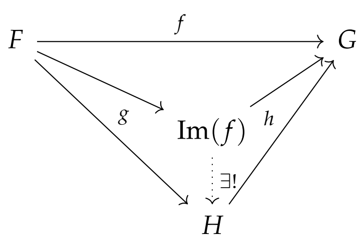
$$PIm(f) := \ker(G \to PCok(f))$$

2. Let $f : F \to G$ a sheaf morphism. The sheaf image is defined as

$$SIm(f) := \ker(G \to SCok(f))$$

From general category theory we have the universal property of an image (which holds for both definitions of images in their respective category):

Proposition 36. *(Universal property of an image)*
For each (pre)sheaf H and (pre)sheaf morphisms $g : F \to H$ and $h : H \to G$ s.t. $f = h \circ g$ there is a unique morphism $\mathrm{Im}(f) \to H$ s.t. the following diagram commutes:



Definition 37. (Exact sequence)
Let

$$\ldots \longrightarrow F \xrightarrow{\ f\ } G \xrightarrow{\ g\ } H \longrightarrow \ldots$$

a sequence of presheaves.
The sequence is called exact at G as sequence of presheaves if $PIm(f) = \ker(g)$ (as subpresheaves of G).
The sequence is called exact at G as sequence of sheaves if $SIm(f) = \ker(g)$ (as subsheaves of G).
The sequence is exact as a sequence of (pre)sheaves if it is exact as sequence of (pre)sheaves at all points.

Theorem 38. *1. The sequence $F \to G \to H$ is an exact sequence of presheaves iff $\forall U \subseteq X$ open the sequence $F(U) \to G(U) \to H(U)$ is exact (as a sequence of abelian groups).*

2. The sequence $F \to G \to H$ is an exact sequence of presheaves iff $\forall c \in X$ the sequence $F_x \to G_x \to H_x$ is exact (as a sequence of abelian groups).

3. If $F \to G \to H$ is a sequence of sheaves, which is exact as sequence of presheaves, it is exact as sequence of sheaves too.

Proof. 1. $F \xrightarrow{f} G \xrightarrow{g} H$ is an exact sequence of presheaves
$\Leftrightarrow \ker g = PIm(f)$

$\Leftrightarrow$ For all $U \subseteq X$ open: $\ker g(U) = \overbrace{\ker(G(U) \to \frac{G(U)}{\operatorname{Im} f(U)})}^{=\operatorname{Im} f(U)}$
$\Leftrightarrow$ For all $U \subseteq X$ open: $F(U) \to G(U) \to H(U)$ is exact.

2. The same steps as for 1. but on the level of stalks is used:
$F \xrightarrow{f} G \xrightarrow{g} H$ is an exact sequence of sheaves
$\Leftrightarrow \ker g = SIm(f)$

$\Leftrightarrow$ For all $x \in X$: $\ker g_x = \overbrace{\ker(G_x \to \frac{G_x}{\operatorname{Im} f_x})}^{=\operatorname{Im} f_x}$
$\Leftrightarrow$ For all $x \in X$: $F_x \to G_x \to H_x$ is exact.

3. $\ker g = PIm(f) \Rightarrow$ For all $x \in X$:

$$\begin{aligned}(\ker g)_x &= (PIm(f))_x \\ &= \ker(G_x \to PCok(f)_x) \\ &\overset{??}{=} \ker(G_x \to (SCok(f))_x) \\ &= (SIm(f))_x\end{aligned}$$

□

With the last theorem and a few theorems about exact sequences in **Ab** the category of abelian groups, we get the following corollary:

Corollary 39. *In the categories* $\mathbf{Shv}_X$ *and* $\mathbf{Presh}_X$ *we have:*

a) $0 \to F \xrightarrow{f} G$ *is exact iff* f *is an monomorphism.*

b) $F \xrightarrow{f} G \to 0$ *is exact iff* f *is an epimorphism.*

c) For any morphism $f : F \to G$ *the sequence* $0 \to \ker f \to F \to G \to Cok(f) \to 0$ *is exact.*

d) $0 \to F \xrightarrow{\mathrm{f}} \mathrm{G} \xrightarrow{\mathrm{g}} \mathrm{H}$ *is exact* $\Leftrightarrow$ f *is a kernel of* g*.*

e) $\mathrm{F} \xrightarrow{\mathrm{f}} \mathrm{G} \xrightarrow{\mathrm{g}} \mathrm{H} \to 0$ *is exact* $\iff$ g *is a cokernel of* f*.*

As last concept we will define the notion of an exact functor which will be needed in the definition of (co)homology.

Definition 40. Let $T : \mathcal{C} \to \mathcal{D}$ a covariant functor between abelian categories $\mathcal{C}$ and $\mathcal{D}$. T is called exact if for each short exact sequence $0 \to F \to G \to H \to 0$ in $\mathcal{C}$ the sequence $0 \to TF \to TG \to TH \to 0$ is exact.
It is called left exact if it only preserves the zero at the left side and right exact if it only preserves the zero at the right side.

Remark 41. From general category theory we know that T is exact iff it preserves all exact sequences (not necessarily short ones).

Theorem 42. *a) The inclusion functor* $\textbf{Shv}_X \hookrightarrow \textbf{Presh}_X$ *is left exact.*

b) The functor "sections over U" $\textbf{Presh}_X \to \textbf{Ab}$, $F \mapsto F(U)$, *for an* $U \subseteq X$ *open, is exact.*

c) The functor $\Gamma(-, U) : \textbf{Shv}_X \to \textbf{Ab}$, $F \to \Gamma(F, U)$ *is left exact.*

Proof. a) Let $0 \to F \xrightarrow{f} G \xrightarrow{g} H \to 0$ exact in $\textbf{Shv}_X$. Then f is a kernel of g in $\textbf{Shv}_X$ by Corollary **??**. As the kernels in the categories are the same, f is also a kernel of g in $\textbf{Presh}_X$ and by Corollary **??** $0 \to F \xrightarrow{f} G \xrightarrow{g} H$ is an exact sequence of presheaves.

b) This is precisely the first part of theorem **??**.

c) This follows by composition of a) and c).

□

Remark 43. You could also proof, that the sheafification functor ΓL is exact. The proof uses the fact that the direct limit is exact and therefore from an exact sequence of presheaves we get an exact sequence on the level of stalks. But the stalks of the presheaf and its sheafification are naturally isomorphic (last talk) so we get a exact sequence of the stalks of the sheafification and using our previous results the sequence of sheaves is then exact.

Using this terminology we can give a concise summary of our examples regarding the exponential sheaf morphism:

Example 44. (The exponential sheaf sequence)
The sequence

$$0 \to \mathbb{Z}_\Omega \to \mathcal{O}_\Omega \xrightarrow{\exp} \mathcal{O}_\Omega^* \to 1$$

is exact as a sequence of sheaves (but not as a sequence of presheaves).